Radiation Transport

Outline

• Definitions, assumptions and terminology
  • Equilibrium limit
• Radiation transport equation
  • Characteristic form & formal solution
  • Material radiative properties
    • Absorption, emission, scattering
• Coupled systems
  • LTE / non-LTE
• Line radiation
  • Line shapes
  • Redistribution
  • Material motion
• Solution methods
  • Transport operators
  • 2-level & multi-level atoms
  • Escape factors
Basic assumptions

Classical / Semi-classical description –

• Radiation field described by either specific intensity $I_\nu$ or the photon distribution function $f_\nu$
• Unpolarized radiation
• Neglect index of refraction effects ($n\approx 1$, $\omega >> \omega_p$)
  ➔ Photons travel in straight lines
• Neglect true scattering (mostly)
• Static material (for now)
  ➔ Single reference frame
Radiation Description

Macroscopic - specific intensity $I_v$
- energy per (area $\times$ solid angle $\times$ time) within the frequency range $(\nu, \nu + d\nu)$

$$dE = I_v(\vec{r}, \Omega, t)(\vec{n} \cdot \vec{\Omega}) dA d\Omega dv dt$$

- $dE$ = energy crossing area $dA$ within $(d\Omega dv dt)$

Microscopic - photon distribution function

$$dE = \sum_{spins} (hv)f_v(\vec{r}, \vec{p}, t) \frac{d^3\vec{x}d^3\vec{p}}{h^3}$$

$$I_v = 2 \frac{hv^3}{c^2} f_v(\vec{r}, \vec{p}, t) \quad \vec{p} = \frac{hv}{c} \vec{\Omega}$$
Angular Moments

0\textsuperscript{th} moment  \quad \quad J_v = \frac{1}{4\pi} \int I_v \, d\Omega \quad = \text{energy density} \times c/4\pi

1\textsuperscript{st} moment  \quad \quad H_v = \frac{1}{4\pi} \int nI_v \, d\Omega \quad = \text{flux} \times 1/4\pi

2\textsuperscript{nd} moment  \quad \quad K_v = \frac{1}{4\pi} \int nnI_v \, d\Omega \quad = \text{pressure tensor} \times c/4\pi

For isotropic radiation, $K_v$ is diagonal with equal elements:

$$K_v = -\frac{1}{3} J_v I \quad (P = \frac{1}{3} E)$$

In this case, radiation looks like an ideal gas with $\gamma = 4/3$
Thermal Equilibrium

Intensity: Planck function

\[ B_v = 2 \frac{h \nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT_r}} - 1} \]

Distribution function: Bose-Einstein

\[ f_v = \frac{1}{e^{\frac{h\nu}{kT_r}} - 1} \]

Energy density

\[ E_{rad} (T_r) = \frac{4\pi}{c} \int_0^\infty B_v (T_r) \, dv = aT_r^4 \]

For \( T_e = T_r \) and \( n_H = 10^{23} \text{ cm}^{-3} \) : \( E_{rad} = E_{\text{matter}} \iff T \approx 300 \text{ eV} \)

**LTE** (Local Thermodynamic Equilibrium) - particles have thermal distributions \( (T_e, T_i) \) photon distribution can be arbitrary
Radiation Transport Equation

\[
\frac{1}{c} \frac{\partial I_v}{\partial t} + \tilde{\Omega} \cdot \nabla I_v = -\alpha_v I_v + \eta_v
\]

\( \alpha_v \) = absorption coefficient (fraction of energy absorbed per unit length)

\( \eta_v \) = emissivity (energy emitted per unit time, volume, frequency, solid angle)

- Boltzmann equation for the photon distribution function:

\[
\frac{1}{c} \frac{\partial f_v}{\partial t} + \tilde{\Omega} \cdot \nabla f_v = \left( \frac{\partial f_v}{\partial t} \right)_{\text{coll}}
\]

\( I_v = 2h\nu \left( \frac{h\nu}{c} \right)^2 f_v \)

- The LHS describes the flow of radiation in phase space
- The RHS describes absorption and emission
  - Absorption & emission coefficients depend on atomic physics
  - Photon # is not conserved (except for scattering)
- Photon mean free path \( \lambda_v = 1/\alpha_v \)
Characteristic Form

Define the source function $S_v$ and optical depth $\tau_v$:

$$S_v = \frac{\eta_v}{\alpha_v} = B_v \text{ in LTE}$$

$$d\tau_v = \alpha_v \, ds$$

Along a characteristic, the radiation transport equation becomes

$$\frac{dI_v}{d\tau_v} = -I_v + S_v \implies I_v(\tau_v) = I_v(0) e^{-\tau_v} + \int_0^{\tau_v} e^{-(\tau_v - \tau_v')} S_v(\tau_v') d\tau_v'$$

This solution is useful when material properties are fixed, e.g. postprocessing for diagnostics

Important features:

- Explicit non-local relationship between $I_v$ and $S_v$
- Escaping radiation comes from depth $\tau_v \approx 1$
- Implicit $S_v(I_v)$ dependence comes from radiation / material coupling

Self-consistently determining $S_v$ and $I_v$ is the hard part of radiation transport
Limiting Cases

Optically-thin $\tau_{\nu}<<1$ (viewed in emission):

$$I_{\nu} = \int_{0}^{\tau_{\nu}} S_{\nu} d\tau' = \int_{0}^{dx} \eta_{\nu} dx' \rightarrow \eta_{\nu} dx$$

$I_{\nu}$ reflects spectral characteristics of emission, independent of absorption

Optically-thick $\tau_{\nu}>>1$:

$$I_{\nu}(\tau_{\nu}) = \int_{0}^{\tau_{\nu}} e^{-(\tau_{\nu}-\tau'_{\nu})} S_{\nu}(\tau'_{\nu}) d\tau'_{\nu} \rightarrow S_{\nu}(\tau_{\nu})$$

$I_{\nu}$ reflects spectral characteristics of $S_{\nu}$ (over \sim last optical depth)

Negligible emission (viewed in absorption):

$$I_{\nu}(\tau_{\nu}) = I_{\nu}(0) e^{-\tau_{\nu}}, \quad \tau_{\nu} = \int_{0}^{dx} \alpha_{\nu} dx' \rightarrow \alpha_{\nu} dx$$

$I_{\nu} / I_{\nu}(0)$ reflects spectral characteristics of absorption
Example – Flux from a uniform sphere

Kr @ T = 200 eV, ρ = 0.01 g/cc, LTE

Δr = 0.001 cm
Example – Flux from a uniform sphere

Kr @ T = 200 eV, ρ = 0.01 g/cc, LTE

Δr = 0.1 cm
Example – Flux from a uniform sphere

Kr @ T = 200 eV, ρ = 0.01 g/cc, LTE

Δr = 10.0 cm
Absorption / emission coefficients

Macroscopic description – energy changes
• Energy removed from radiation passing through material of area $dA$, depth $ds$, over time $dt$

$$dE = -\alpha_\nu I_\nu dA ds d\Omega d\nu dt$$

• Energy emitted by material

$$dE = \eta_\nu dA ds d\Omega d\nu dt$$

Microscopic description – radiative transitions
• Absorption and emission coefficients are constructed from atomic populations $y_i$ and cross sections $\sigma_{ij}$:

$$\alpha_\nu = \sum_{i<j} \sigma_{v,ij} (y_i - \frac{g_i}{g_j} y_j), \quad \eta_\nu = \frac{2\hbar^3}{c^2} \sum_{i<j} \sigma_{v,ij} \frac{g_i}{g_j} y_j$$
Bound-bound Transitions

Probability (per unit time) of
• Spontaneous emission: \( A_{21} \)
• Absorption: \( B_{12} \)
• Stimulated emission: \( B_{21} \)

\( A_{21}, B_{12}, B_{21} \) are Einstein coefficients

\[
g_1 B_{21} = g_2 B_{12}, \quad A_{21} = \frac{2h\nu_0^3}{c^2} B_{21}
\]

Line profile \( \phi(\nu) \) measures probability of absorption

\[
\int_0^\infty \phi(\nu) d\nu = 1
\]

Transition rate from level 1 to level 2

\[
R_{12} = B_{12} \overline{J}, \quad \overline{J} = \int_0^\infty J \phi(\nu) d\nu
\]

(assuming that the linewidth \( \ll \Delta E \))
Cross section for absorption

\[ \sigma(v) = \frac{\hbar \nu}{4\pi} B_{21} \phi(v) = \frac{\pi e^2}{mc} f_{12} \phi(v) \]

\[ f_{12} = \text{oscillator strength} \]
\[ \frac{\pi e^2}{mc} = 0.02654 \text{ cm}^2/\text{s} \]

Oscillator strength \( f_{12} \) relates the quantum mechanical result to the classical treatment of a harmonic oscillator

- Strong transitions have \( f \sim 1 \)
- Sum rule \( \sum_j f_{ij} = Z \) (# of bound electrons)

Absorption and emission coefficients:

Absorption

\[ \alpha_v = n_1 \frac{\pi e^2}{mc} f_{12} \phi(v) \left[ 1 - \frac{g_1 n_2}{g_2 n_1} \right] \]

absorption – stimulated emission

Spontaneous emission

\[ \eta_v = \left( \frac{2\hbar \nu^3}{c^2} \right) n_2 \frac{\pi e^2}{mc} f_{12} \phi(v) \]

For now we are assuming that absorption and emission have the same line profile
Bound-free absorption

Absorption cross section from state of principal quantum number \( n \) and charge \( Z \)

\[
\sigma_{bf} = 7.91 \cdot 10^{-18} \, n g_{bf} \left( \frac{\nu_0}{\nu} \right)^3 \left( 1 - e^{-\frac{h\nu_{0}}{kT_e}} \right) \text{cm}^2
\]

Gaunt factor \( h\nu_0 = \) threshold energy

Free-free absorption

Absorption cross section per ion of charge \( Z \)

\[
\sigma_{ff} = 3.69 \cdot 10^8 \, \frac{Z^2}{\nu^3 \sqrt{T_e}} \, g_{ff} \, n_e \left( 1 - e^{-\frac{h\nu_{0}}{kT_e}} \right) \text{cm}^2
\]

Gaunt factor

The term \( 1 - e^{-\frac{h\nu_{0}}{kT_e}} \) accounts for stimulated emission
Scattering

Interaction in which the photon energy is (mostly) conserved (i.e. not converted to kinetic energy)

Examples:
- Scattering by bound electrons – Rayleigh scattering
  - important in atmospheric radiation transport
- Scattering by free electrons – Thomson / Compton scattering

\[ \sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{m_e c^2} \right)^2 = 6.65 \times 10^{-25} \text{ cm}^2 \]

\[ hv \ll m_e c^2 \]

Note: frequency shift from scattering is \( \sim \frac{hv}{m_e c^2} \)

Doppler shift from electron velocity is \( \sim \sqrt{\frac{2kT}{m_e c^2}} \)

For most laboratory plasmas, these types of scattering are negligible

Note: X-ray Thomson scattering (off ion acoustic waves and plasma oscillations) can be a powerful diagnostic for multiple plasma parameters \((T_e, T_i, n_e)\)
Radiation transport equation with scattering (and frequency changes):

\[
\frac{1}{c} \frac{\partial I_v}{\partial t} + \tilde{\Omega} \cdot \nabla I_v = -\alpha_v I_v + \eta_v + \sigma_v \int_0^\infty dV' \int d\Omega' \frac{d\Omega'}{4\pi} \left[ -R(V',\Omega';V,\Omega) \frac{V}{V'} I_v (1 + f'_V) + R(V,\Omega;V',\Omega') I_{v'} (1 + f_{v'}) \right]
\]

The redistribution function \( R \) describes the scattering of photons \((\nu, \Omega) \rightarrow (\nu', \Omega')\)

Neglecting frequency changes, this simplifies to

\[
\frac{1}{c} \frac{\partial I_v}{\partial t} + \tilde{\Omega} \cdot \nabla I_v = -(\alpha_v + \sigma_v) I_v + \eta_v + \sigma_v \int d\Omega' I_v (\tilde{\Omega}') g(\tilde{\Omega} \cdot \tilde{\Omega}') \\
\rightarrow -(\alpha_v + \sigma_v) I_v + \eta_v + \sigma_v J_v 
\]

for isotropic scattering

Scattering contributes to both absorption and emission terms (and may be denoted separately or included in \( \alpha_v \) and \( \eta_v \)).

stimulated scattering
Effective scattering

Photons also “scatter” by e.g. resonant absorption / emission

Upper level 2 can decay
a) radiatively $A_{21}$
b) collisionally $n_e C_{21}$

The fraction $\epsilon \approx n_e C_{21} / A_{21}$ of photons are destroyed / thermalized

The fraction $(1-\epsilon)$ of photons are “scattered”
- energy changes only slightly (mostly Doppler shifts)
- undergo many “scatterings” before being thermalized

Note: $\epsilon \ll 1$ is the condition for a strongly non-LTE transition and is easily satisfied for low density or high $\Delta E$!
Population distribution

**LTE:** Saha-Boltzmann equation

- Excited states follow a Boltzmann distribution
  \[
  y_i = \frac{g_i}{g_j} e^{-\frac{(\epsilon_i - \epsilon_j)}{T_e}}
  \]
  \(\epsilon_i = \) energy of state \(i\)
  \(T_e = \) electron temperature

- Ionization stages obey the Saha equation
  \[
  \frac{N_q}{N_{q+1}} = \frac{1}{2} \frac{n_e}{U_{q+1}} \left( \frac{\hbar^2}{2\pi m_e T_e} \right)^{3/2} e^{-\frac{(\epsilon_0^q - \epsilon_0^q)}{T_e}}
  \]
  \(N_q = \sum_{i \in q} y_i e^{-\frac{(\epsilon_i - \epsilon_0^q)}{T_e}}\) number density of charge state \(q\)
  \(U_q = \sum_{i \in q} g_i e^{-\frac{(\epsilon_i - \epsilon_0^q)}{T_e}}\) partition function of charge state \(q\)

**NLTE:** Collisional-radiative model

- Calculate populations by integrating rate equations
  \[
  \frac{dy}{dt} = Ay, \quad A_{ij} = C_{ij} + R_{ij} + (\text{other})_{ij}
  \quad C_{ij} = n_e \int v \sigma_{ij}(v) f(v) dv
  \]
  \[
  R_{ij} = \int \sigma_{ij}(v) J(v) \frac{dv}{hv}, \quad R_{ji} = \int \sigma_{ij}(v) \left[ J(v) + \frac{2hv^3}{c^2} \right] e^{-\frac{hv}{kT}} \frac{dv}{hv}
  \]
Summary of absorption / emission coefficients

- Summed over populations and radiative transitions:

\[ \alpha_v = \sum_{ij} y_i \left\{ \frac{\pi e^2}{mc^2} f_{ij} \phi_{ij}(v) \left( 1 - \frac{g_i y_j}{g_j y_i} \right) + \left[ \sigma_{ij}^{bf}(v) + n_e \bar{\sigma}_{ij}^{ff}(v) \right] (1 - e^{-h \nu / k T_e}) \right\} \]

\[ \eta_v = \frac{2h \nu^3}{c^2} \sum_{ij} y_j \left\{ \frac{\pi e^2}{mc^2} g_i f_{ij} \phi_{ij}(v) + \left[ \sigma_{ij}^{bf}(v) + n_e \bar{\sigma}_{ij}^{ff}(v) \right] e^{-h \nu / k T_e} \right\} \]

- In NLTE, the source function has a complex dependence on plasma parameters and on the radiation spectrum:

\[ S_v = S_v(n_e, T_e; y_i(J_v)) \]

- In LTE, absorption and emission spectra are complex but the source function obeys Kirchoff’s law:

\[ S_v = B_v(T_e) \]
Opacity & mean opacities

Opacity = absorption coefficient / mass density \[ \kappa_v = \alpha_v / \rho \]

Rosseland mean opacity:
- emphasizes transmission
- includes scattering
- appropriate for average flux

Planck mean opacity:
- emphasizes absorption
- no scattering
- appropriate for energy exchange

\[ \frac{1}{\kappa_R} = \int_0^\infty d\nu \frac{1}{\kappa_v} \frac{dB_v}{dT} \]
\[ \kappa_P = \int_0^\infty d\nu \kappa_v B_v \]

Kr @ T = 200 eV, \( \rho = 0.01 \) g/cc, LTE
Example – Hydrogen ($T_e = 2 \text{ eV}, n_e = 10^{14} \text{ cm}^{-3}$)

absorption coefficient

![Graph showing absorption coefficient against photon energy for Hydrogen with $n_e = 10^{14} \text{ cm}^{-3}$ and $T_e = 2 \text{ eV}$, with LTE and NLTE conditions.](image-url)
Hydrogen, again ($T_e = 2 \text{ eV}, n_e = 10^{14} \text{ cm}^{-3}$)

emissivity

source function

emissivity (erg/cm$^3$/s/eV/ster)

source function (erg/cm$^2$/s/eV/ster)

photon energy (eV)

photon energy (eV)
Flux from a uniform sphere revisited

Kr @ T = 200 eV, ρ = 0.01 g/cc, LTE / NLTE

Δr = 0.001 cm
Flux from a uniform sphere revisited

Kr @ T = 200 eV, ρ = 0.01 g/cc, LTE / NLTE

Δr = 0.1 cm
Flux from a uniform sphere revisited

Kr @ T = 200 eV, ρ = 0.01 g/cc, LTE / NLTE

Δr = 10.0 cm
Flux from a uniform sphere - Summary

- “Black-body” emission requires large optical depths.
- Large optical depths \(\Rightarrow\) high radiation fields \(\Rightarrow\) LTE conditions.
- Boundaries introduce non-uniformities through radiation fields.
- The radiation field will also change the material temperature.

Conditions do not remain uniform in the presence of radiation transport and boundaries.
Example – Hydrogen Ly-α

Ly-α emission from a uniform plasma

- $T_e = 1 \text{ eV}, n_e = 10^{14} \text{ cm}^{-3}$
- Moderate optical depth $\tau \sim 5$
- Viewing angles $90^\circ$ and $10^\circ$
  show optical depth broadening

Example – Hydrogen Ly-α
Example – Hydrogen Ly-α

Ly-α emission from a plasma with uniform temperature and density

- $T_e = 1$ eV, $n_e = 10^{14}$ cm$^{-3}$
- Self-consistent solution displays effects of
  - Radiation trapping / pumping
  - Non-uniformity due to boundaries

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**Specific Intensity (cgs)**

- Photon energy w.r.t. line center (eV)
- Position (cm)
- Fractional population

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**Example**: Hydrogen Ly-α emission from a plasma with uniform temperature and density.

- $T_e = 1$ eV, $n_e = 10^{14}$ cm$^{-3}$
- Self-consistent solution displays effects of
  - Radiation trapping / pumping
  - Non-uniformity due to boundaries
Coupled systems – or –
What does “Radiation Transport” mean?

The system of equations and emphasis varies with the application

For laboratory plasmas, these two sets are most useful -

**LTE / energy transport** :
- Coupled to energy balance

\[
\frac{dE_m}{dt} = 4\pi \int \alpha_v (J_v - S_v) \, dv
\]

\[
\alpha_v = \alpha_v(T_e) \quad , \quad S_v = B_v(T_e)
\]

- Indirect radiation-material coupling through energy/temperature
- Collisions couple all frequencies locally, independent of \( J_v \)
- Solution methods concentrate on non-local aspects

**NLTE / spectroscopy** :
- Coupled to rate equations

\[
\frac{dy}{dt} = Ay \quad , \quad A_{ij} = A_{ij}(T_e, J_v)
\]

\[
S_v = \frac{2\hbar \nu^3}{c^2} S_{ij} \quad , \quad S_{ij} \approx a + b \, \bar{J}_{ij}
\]

- Direct coupling of radiation to material
- Collisions couple frequencies over narrow band (line profiles)
- Solution methods concentrate on local material-radiation coupling
- Non-local aspects are less critical
Line Profiles

Line profiles are determined by multiple effects:
- Natural broadening \((A_{12})\)
- Collisional broadening \((n_e, T_e)\)
- Doppler broadening \((T_i)\)
- Stark effect (plasma microfields)

\[
\phi(v) = \frac{1}{\Delta \nu_D \sqrt{\pi}} H(a, x)
\]

\[
a = \frac{\Gamma}{4\pi \Delta \nu_D}
\]

\[
\Delta \nu_D = \frac{v_0}{c} \sqrt{\frac{2kT_i}{m_i}}
\]

\[
H(a, x) = \frac{a \pi}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(x-y)^2 + a^2} dy
\]

\[
\phi(v) = \frac{\Gamma/4\pi^2}{(v - v_0)^2 + (\Gamma/4\pi)^2}
\]

\(\Gamma = \) destruction rate

Gaussian, Voigt and Lorentzian Profiles

- Lorentzian
- Gaussian
- Complex

\(\phi(v)\) vs \((v-v_0)/\Delta\)
Redistribution

The emission profile $\psi_v$ is determined by multiple effects:
- collisional excitation $\Rightarrow$ natural line profile (Lorentzian)
- photo excitation + coherent scattering
- photo excitation + elastic scattering $\Rightarrow$ absorption profile
- Doppler broadening

This is described by the redistribution function $R(v,v')$

$$\int_0^\infty R(v,v')\, dv = \phi(v') \quad , \quad \psi(v) = \int_0^\infty R(v,v')J(v')\, dv' \left/ \int_0^\infty \phi(v')J(v')\, dv' \right.$$  

Complete redistribution (CRD): $\psi_v = \phi_v$

Doppler broadening is only slightly different from CRD, while coherent scattering gives $R(v,v') = \phi(v)\delta(v - v')$

A good approximation for partial redistribution (PRD) is often

$$R(v,v') = (1 - f)\phi(v')\phi(v) + f R_{II}(v,v')$$

where $f$ ($<<1$ for X-rays) is the ratio of elastic scattering and de-excitation rates, $R_{II}$ includes coherent scattering and Doppler broadening.
Hydrogen Ly-α w/ Partial Redistribution

Ly-α emission from a plasma with uniform temperature and density

- $T_e = 1$ eV, $n_e = 10^{14}$ cm$^{-3}$
- Optical depth $\tau \sim 5$
- Voigt parameter $a \sim 0.0003$

![Graph showing specific intensity and n=2 fractional population vs. photon energy and position.](image)
Material Velocity

The discussion so far applies in the reference frame of the material. Doppler shifts matter for line radiation when $v/c \sim \Delta E/E$.

If velocity gradients are present, either

a) Transform the RTE into the co-moving frame - or –

b) Transform material properties into the laboratory frame.

Option (a) is complicated (particularly when $v/c \approx 1$)

- see the references by Castor and Mihalas for discussions

Option (b) is relatively simple, but makes the absorption and emission coefficients direction-dependent.
Al sphere w/ uniform expansion velocity

\( T_e = 500 \text{ eV} \)
\( \rho = 10^{-3} \text{ g/cm}^3 \)
\( R_{\text{outer}} = 1.0 \text{ cm} \)
\( R_{\text{inner}} = 0.9 \text{ cm} \)

**optical depth**

**areal flux**

![Graphs showing optical depth and areal flux for different velocities](image)

- **Graph 1:** Optical depth vs. photon energy (eV) for different expansion velocities (v/c).
- **Graph 2:** Areal flux (erg/s/Hz/ster) vs. photon energy (eV) for different expansion velocities (v/c).

- **v/c = 0** (blue)
- **v/c = 0.003** (red)
- **v/c = 0.01** (black)
- **v/c = 0.03** (green)
Al sphere w/ uniform expansion velocity

![Graph showing areal flux (erg/s/Hz/ster) vs photon energy (eV) with different expansion velocities (v/c = 0, v/c = 0.003, v/c = 0.01, v/c = 0.03). Peaks at He-α and Ly-α wavelengths.]
2-Level Atom

Rate equation for two levels in steady state:

\[ n_1(B_{12}J_{12} + C_{12}) = n_2(A_{21} + B_{21}J_{12} + C_{21}) \]

\[ J_{12} = \int_0^\infty J_v \phi_{12}(v) dv, \quad C_{12} = \frac{g_2}{g_1} e^{-h\nu_0/kT} C_{21} \]

Absorption / Emission:

\[ \alpha_v = \frac{h\nu}{4\pi} (n_1 B_{12} - n_2 B_{21}) \phi_{12}(v), \quad \eta_v = \frac{h\nu}{4\pi} n_2 A_{21} \phi_{12}(v) \]

Source Function:

\[ S_v = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = (1 - \varepsilon)J_{12} + \varepsilon B_v, \quad \frac{\varepsilon}{1 - \varepsilon} = \frac{C_{21}}{A_{21}} \left(1 - e^{-h\nu_0/kT}\right) \]

\( S_v \) is independent of frequency and linear in \( J \)

- solution methods exploit this dependence
A Popular Solution Technique

For a single line, the system of equations can be expressed as

\[ \frac{dI_v}{d\tau_v} = -I_v + S_v \Rightarrow I_v = \tilde{\lambda}_v S_v(\bar{J}) \]

\[ S_v = a_v + b_v \bar{J} \text{, } \bar{J} = \frac{1}{4\pi} \int d\Omega \int_0^\infty I_v \phi(v) dv \]

where the lambda operator \( \tilde{\lambda}_v \) and the source function depend on the populations through the absorption and emission coefficients.

A numerical solution for \( \bar{J} \) uniquely specifies the entire system.

Since \( \tilde{\lambda}_v \) depends on \( \bar{J} \) through the populations, the full system is non-linear and requires iterating solutions of the rate equations and radiation transport equation.

The dependence of \( \tilde{\lambda}_v \) on \( \bar{J} \) is usually weak, so convergence is rapid.
Solution technique for the linear source function:

\[
I_v = \tilde{\lambda}_v \left[ a_v + b_v J \right]
\]

\[\Rightarrow \quad J = \frac{1}{4\pi} \int d\Omega \int_0^\infty \tilde{\lambda}_v \left[ a_v + b_v J \right] \phi(v) dv\]

\[\Rightarrow \quad J = \left[ 1 - \tilde{\Lambda} \right]^{-1} \frac{1}{4\pi} \int d\Omega \int_0^\infty \tilde{\lambda}_v a_v \phi(v) dv\]

\[\tilde{\Lambda} = \frac{1}{4\pi} \int d\Omega \int_0^\infty \tilde{\lambda}_v b_v \phi(v) dv\]

This linear system can (in principle) be solved directly for \( J \) and in 1D this is very efficient.

The (angle,frequency)-integrated operator \( \tilde{\Lambda} \) includes \( 1 - e^{-\tau_v} \) factors, so \( \left[ 1 - \tilde{\Lambda} \right]^{-1} \) amplifies \( J \) (radiation trapping).

Efficient solution methods approximate key parts of \( \tilde{\Lambda} \):
- NLTE – local frequency scattering
- LTE (linear in \( \Delta T_e \)) – non-local coupling
Multi-Level Atom

For multiple lines, the source function for each line can be put in the two-level form – ETLA (Extended Two Level Atom) – or the full source function can be expressed in the following manner:

\[ S_v = \frac{\eta_v^c + \sum \eta_v^\ell (\bar{J}_\ell \phi_v^\ell)}{\alpha_v^c + \sum \alpha_v^\ell (\bar{J}_\ell \phi_v^\ell)} \]

Solve for each \( \bar{J}_\ell \) individually (if coupling between lines is not important) or simultaneously.

(Complete) linearization – expand \( S_v \) in terms of \( \Delta \bar{J}_\ell \)

\[ S_v = S_v(\bar{J}_\ell^0) + \sum \frac{\partial S_v}{\partial \bar{J}_\ell} (\bar{J}_\ell - \bar{J}_\ell^0) \]

and solve as before.

Partial redistribution usually converges at a slightly slower rate.
Numerical methods need to fulfill 2 requirements

1. Accurate formal transport solution which is
   • conservative,
   • non-negative
   • 2nd order (spatial) accuracy (diffusion limit as $\tau >> 1$)
   • causal (+ efficient)
   
   Many options are available – each has advantages and disadvantages

2. Method to converge solution of coupled implicit equations
   • Multiple methods fall into a few classes
     – Full nonlinear system solution
     – Accelerated transport solution
     – Incorporate transport information into other physics
   • Optimized methods are available for specific regimes, but no single method works well across all regimes
Method #1 – Source (or lambda) iteration

1. Evaluate source function
2. Formal solution of radiation transport equation
3. Use intensities to evaluate temperature / populations

Advantages –
Simple to implement
Independent of formal transport method

Disadvantages –
Can require many iterations: \# iterations \( \sim \tau^2 \)
False convergence is a problem for \( \tau \gg 1 \)
Hydrogen Lyman-α revisited

- Source iteration (green curves) approaches self-consistent solution slowly
- Linearization achieves convergence in 1 iteration

\[ S_{ij} = a + b \overline{J}_{ij}, \overline{J}_{ij} = \int J_{\nu} \phi_{\nu} \, d\nu \]

\[ \overline{J} \]

\[ n=2 \text{ fractional population} \]

\[ \text{specific intensity (cgs)} \]

\[ \text{photon energy w.r.t. line center (eV)} \]

\[ \text{position (cm)} \]
Method #2 – Monte Carlo

Formal solution method –
1) Sample emission distribution in (space, frequency, direction) to create “photons”
2) Track “photons” until they escape or are destroyed

Advantages –
  Works well for complicated geometries
  Not overly constrained by discretizations ➔ does details very well

Disadvantages –
  Statistical noise improves slowly with # of particles
  Expense increases with optical depth
  Iterative evaluation of coupled system is not possible / advisable
  Semi-implicit nature requires careful timestep control

Convergence –
  A procedure that transforms absorption/emission events into effective scatterings (Implicit Monte Carlo) provides a semi-implicit solution

Symbolic IMC provides a fully-implicit solution at the cost of solving a single mesh-wide nonlinear equation
Method #3 – Discrete Ordinates (S\textsubscript{N})

**Formal solution method** –
Discretization in angle converts integro-differential equations into a set of coupled
differential equations (provides lambda operator)

**Advantages** –
Handles regions with $\tau<<1$ and $\tau>>1$ equally well
Modern spatial discretizations achieve the diffusion limit
Deterministic method can be iterated to convergence

**Disadvantages** –
Ray effects due to preferred directions
angular profiles become inaccurate well before angular integrals
Required # of angles in 2D/3D can become enormous
Discretization in 7 dimensions requires large computational resources

**Convergence** –
Effective solution algorithms exist for both LTE and NLTE versions [6] –
e.g. LTE – synthetic grey transport (or diffusion)
   NLTE – complete linearization (provides linear source function)
      + accelerated lambda iteration in 2D/3D

(Note: this applies to all deterministic methods)
Method #4 – Escape factors

Escape factor $p_e$ is used to eliminate radiation field from net radiative rate

$$y_j B_{ji} \overline{J} - y_i B_{ij} \overline{J} = y_j A_{ji} p_e$$

Equivalent to incorporating a (partial) lambda operator into the rate equations

→ combines the formal solution + convergence method

Advantages –

Very fast – no transport equation solution required
Can be combined with other physics with no (or minimal) changes

Disadvantages –

Details of transport solution are absent
Escape factors depend on line profiles, system geometry
Iterative improvement is possible, but usually not worthwhile
Escape factor is built off the single flight escape probability $e^{-\tau_v}$

$$p_e = \frac{1}{4\pi} \int d\Omega \int_0^\infty e^{-\tau_v} \phi(v) dv$$

Iron’s theorem – this gives the correct rate on average
(spatial average weighted by emission)

$$\left\langle 1 - \frac{\bar{J}}{S} \right\rangle = \langle p_e \rangle$$

Note that the optical depth depends on the line profile, plus continuum

Asymptotic expressions for large optical depth:
- Gaussian profile
- Voigt profile

$$p_e \approx \frac{1}{4\pi} \sqrt{\ln(\tau/\sqrt{\pi})} \quad \quad p_e \approx \frac{1}{3} \sqrt{a/\tau}$$

Evaluating $p_e$ can be complicated by overlapping lines, Doppler shifts, etc.

Many variations and extensions exist in a large literature
References


